

# Geometry of Numbers

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**Abstract.** We develop a global cohomology theory for number fields by offering topological cohomology groups, an arithmetical duality, a Riemann-Roch theorem, and two types of vanishing theorem. As applications, we study moduli spaces of semi-stable lattices, and introduce non-abelian zeta functions for number fields.<sup>1</sup>

## 1 Global Cohomology

### 1.1 Adelic cohomology groups

Let  $F$  be a number field, and  $\mathcal{O}_F$  its integer ring. Denote  $S := S_{\text{fin}} \cup S_{\infty}$  the set of normalized non-Archimedean and Archimedean places of  $F$ . For all  $v \in S$ , write  $F_v$  the  $v$ -completion of  $F$ ,  $\mathcal{O}_v$  its integer ring.

Denote  $\mathbb{A}$  the adelic ring of  $F$ ,  $GL_n(\mathbb{A})$  the associated general linear group. For each  $g = (g_{\mathfrak{p}}; g_v) \in GL_n(\mathbb{A})$ ,  $\mathfrak{p} \in S_{\text{fin}}, v \in S_{\infty}$ , introduce an auxiliary topological space  $\mathbb{A}^n(g)$  as follows:

(i) Set theoretically,

$$\mathbb{A}^n(g) := \left\{ \mathbf{x} \in \mathbb{A}^n \mid \begin{array}{l} \exists \mathbf{a} \in F^n, \text{ s.t. } \mathbf{x}_v = \mathbf{a}, \quad \forall v \in S_{\infty} \\ g_{\mathfrak{p}} \cdot \mathbf{x}_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^n, g_{\mathfrak{p}} \cdot \mathbf{a} \in \mathcal{O}_{\mathfrak{p}}^n, \quad \forall \mathfrak{p} \in S_{\text{fin}} \end{array} \right\};$$

(ii) Topologically, first introduce a new topological structure on  $\mathbb{A}^n$  by keeping the finite part but altering its metric at  $v \in S_{\infty}$  using the positive definite matrix  $g_{\sigma}^t \cdot g_{\sigma}$  (resp.  $\bar{g}_{\tau}^t \cdot g_{\tau}$ ) when  $v = \sigma$  is real (resp.  $v = \tau$  is complex); then equip  $\mathbb{A}^n(g)$  with the induced topological structure from the embedding  $\mathbb{A}^n(g) \subset \mathbb{A}^n$ .

**Definition 1.** For a matrix idele  $g \in GL_n(\mathbb{A})$ , define the 0-th and the 1-st arithmetical cohomology groups of  $g$  by:

(1) As abstract abelian groups,

$$H^0(F, g) := \mathbb{A}^n(g) \cap K^n, \quad \text{and} \quad H^1(F, g) := \mathbb{A}^n / (\mathbb{A}^n(g) + K^n);$$

(2) As topological spaces,  $H^0(F, g)$  and  $H^1(F, g)$  are equipped with topologies induced from the altered  $\mathbb{A}^n$ .

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<sup>1</sup> Parts of this paper were circulated under the title: *Riemann-Roch, Stability and New Non-Abelian Zeta Functions for Number Fields*, arXiv:math/0007146. The new developments on vanishing theorem and strong stability are added to complete the theory.

**Proposition 1.** *As locally compact topological spaces,  $H^0(F, g)$  is discrete and  $H^1(F, g)$  is compact.*

*Proof.* • **Discreteness of  $H^0$ :** Following Minkowski, via Archimedean places, embed  $F$  into  $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ , where  $r_1$  (resp.  $r_2$ ) denotes the number of real places (resp. complex places). Introduce

$$H_{\text{fin}}^0(F, g) := \left\{ \mathbf{x} \in F^n \mid g_{\mathfrak{p}} \cdot \mathbf{x} \in \mathcal{O}_{\mathfrak{p}}^n, \forall \mathfrak{p} \in S_{\text{fin}} \right\}.$$

By definition, particularly, by paying attention to the infinite components, we conclude that, as abelian groups,  $H^0(F, g) \simeq H_{\text{fin}}^0(F, g)$ , and the natural embedding  $H^0(F, g) \hookrightarrow \mathbb{A}_{\infty}^n$ , via

$$H^0(F, g) \simeq H_{\text{fin}}^0(F, g) \hookrightarrow F^n \xrightarrow{i} \left( \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \right)^n = \mathbb{A}_{\infty}^n,$$

induces a natural discrete topological structure on  $H^0(F, g)$ .

• **Compactness of  $H^1$ :** This is equivalent to the *strong approximation theorem* for adeles. Indeed, if we equip  $\mathbb{A}_{\infty}^n \simeq \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n$  with the twisted metric structure  $\rho(g)$  obtained from the positive definite matrix  $(g_{\sigma}^t \cdot g_{\sigma}; \overline{g_{\tau}}^t \cdot g_{\tau})$ , then  $H^0(F, g) \simeq H_{\text{fin}}^0(F, g)$  can be viewed as a full rank lattice  $\Lambda(g)$  within.

**Proposition 2.** *As locally compact topological groups, we have the following natural isomorphism*

$$H^1(F, g) \simeq \left( \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n, \rho(g_{\infty}) \right) / \Lambda(g).$$

*Proof.* By keeping track on the metrics, it suffices to show that, as abstract groups, we have an isomorphism

$$\mathbb{A}^n / (\mathbb{A}^n(g) + F^n) \simeq \mathbb{A}_{\infty}^n / i(H_{\text{fin}}^0(F, g)).$$

For this, we use the natural morphism  $\phi: \mathbb{A}_{\infty}^n \rightarrow \mathbb{A}^n / (\mathbb{A}^n(g) + F^n)$  induced from the embedding  $\mathbb{A}_{\infty}^n \hookrightarrow \mathbb{A}^n$ . Hence we need to prove that

$$(i) \phi \text{ is surjective;} \quad \& \quad (ii) \text{Ker } \phi = i(H_{\text{fin}}^0(F, g)).$$

The surjectivity (i) is equivalent to  $\mathbb{A}_{\text{fin}}^n \subset \mathbb{A}^n(g) + F^n$ . This can be established using the strong approximation property: Indeed, for any  $g = (g_{\mathfrak{p}}; g_v) \in \mathbb{A}^n$ , there exists a finite set  $S_0 \subset S_{\text{fin}}$  such that  $g_{\mathfrak{q}} \in GL_n(\mathcal{O}_{\mathfrak{q}})$  for  $\mathfrak{q} \notin S_0$ . Moreover, by the strong approximation, there exists  $\mathbf{a} \in K^n$  such that

- (a)  $g_{\mathfrak{p}} \cdot a_{\mathfrak{p}} - \mathbf{a} \in \mathcal{O}_{\mathfrak{p}}^n, \forall \mathfrak{p} \in S_0$ ; and
- (b)  $\mathbf{a} \in \mathcal{O}_{\mathfrak{q}}^n, \forall \mathfrak{q} \notin S_0$ .

(ii) is obtained by examining the definition of  $\mathbb{A}^n(g)$  carefully. Indeed, this is a direct consequence of the conditions for all components, particularly, those at infinity, of the elements in  $H^0(F, g)$ .

## 1.2 Nine diagram: Riemann-Roch axioms

Fix a finite place  $\mathfrak{p}$  of  $F$ . For  $g \in GL_n(F_{\mathfrak{p}})$ , define its *0-th and 1-st cohomology group* by

$$H^0(F_{\mathfrak{p}}, g) := \{x \in F_{\mathfrak{p}}^n : g \cdot x \in \mathcal{O}_{\mathfrak{p}}\} \quad \text{and} \quad H^1(F_{\mathfrak{p}}, g) := F_{\mathfrak{p}}^n / H^0(F_{\mathfrak{p}}, g) + F_{\mathfrak{p}}^n.$$

Elements  $g_1, g_2$  of  $GL_n(F_{\mathfrak{p}})$  are called *equivalent*, denoted by  $g_1 \sim g_2$ , if  $H^0(F_{\mathfrak{p}}, g_1) = H^0(F_{\mathfrak{p}}, g_2)$ . Clearly,  $g_1 \sim g_2$  if and only if there exists  $g \in GL_n(\mathcal{O}_{\mathfrak{p}})$  such that  $g_2 = g \cdot g_1$ . Introduce then the quotient set  $GL_n(F_{\mathfrak{p}}) / GL_n(\mathcal{O}_{\mathfrak{p}})$  and a partial order  $\leq$  on  $GL_n(\mathcal{O}_{\mathfrak{p}}) \backslash GL_n(F_{\mathfrak{p}})$ :

$$[g_1] \leq [g_2] \quad \text{if} \quad H^0(F_{\mathfrak{p}}, g_1) \subset H^0(F_{\mathfrak{p}}, g_2).$$

Globalizing this, we then obtain an equivalence relation  $\sim$  on  $GL_n(\mathbb{A}_{\text{fin}})$  (and more generally on  $GL_n(\mathbb{A}_{\text{fin}}) \times \{g_{\infty}\}$  for a fixed  $g_{\infty} \in GL_n(\mathbb{A}_{\infty})$ ) and a partial order  $\leq$  on  $GL_n(\prod_{\mathfrak{p} \in S_{\text{fin}}} \mathcal{O}_{\mathfrak{p}}) \backslash GL_n(\mathbb{A}_{\text{fin}})$  such that  $g_1 \simeq g_2$  iff  $H_{\text{fin}}^0(F, g_1) = H_{\text{fin}}^0(F, g_2)$  iff there exists  $g \in GL_n(\prod_{\mathfrak{p} \in S_{\text{fin}}} \mathcal{O}_{\mathfrak{p}})$  such that  $g_2 = g \cdot g_1$ ; and

$$[g_1] \leq [g_2] \quad \text{iff} \quad H_{\text{fin}}^0(F, g_1) \subset H_{\text{fin}}^0(F, g_2).$$

Recall that for  $g \in GL_n(\mathbb{A})$ , we have the following 9-diagram with rows and columns exact:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H^0(F, g) & \rightarrow & \mathbb{A}^n(g) & \rightarrow & \mathbb{A}^n(g) / \mathbb{A}^n(g) \cap F^n \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & F^n & \rightarrow & \mathbb{A}^n & \rightarrow & \mathbb{A}^n / F^n \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & F^n / \mathbb{A}^n(g) \cap F^n & \rightarrow & \mathbb{A}^n / \mathbb{A}^n(g) & \rightarrow & H^1(F, g) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0. \end{array}$$

Consequently, if  $g = (g_{\mathfrak{p}}; g_v), g' = (g'_{\mathfrak{p}}; g'_v)$  satisfying  $g_v = g'_v, \forall v \in S_{\infty}$  and  $[(g_{\mathfrak{p}})] \leq [(g'_{\mathfrak{p}})]$ , we have the following exact sequences by noticing that in the above diagram, the middle row remains invariant:

$$H^0(F, g) \hookrightarrow H^0(F, g') \rightarrow \mathbb{A}^n(g') / \mathbb{A}^n(g) \rightarrow H^1(F, g) \twoheadrightarrow H^1(F, g').$$

Note that with  $g_{\infty}$  fixed,  $\mathbb{A}^n(g') / \mathbb{A}^n(g)$  measures only finite places contributions. Namely,  $\deg_{\text{fin}}(g') - \deg_{\text{fin}}(g) = \deg(g') - \deg(g)$  where

$$\deg_{\text{fin}}(g) := \deg_{\text{fin}}(\det g) = \sum_{\mathfrak{p}} \text{ord}_{\mathfrak{p}}(\det g_{\mathfrak{p}}) \cdot N_{\mathfrak{p}} \log p.$$

This then further suggests a way to count  $H^0$  and  $H^1$ . Namely, it should bear the rules of

**Axiom 1.** (*Weak Riemann-Roch for a fixed  $g_\infty$* ) For  $g, g'$  with a fixed  $g_\infty$ ,

$$h^0(F, g') - h^1(F, g') - \deg(g') = h^0(F, g) - h^1(F, g) - \deg(g).$$

**Axiom 2.** (*Riemann-Roch for a fixed  $n$* )

$h^0(F, g) - h^1(F, g) - \deg(g)$  is independent of  $g$ .

Assuming it, then we may write this invariant of  $F$  as  $\Delta(F, n)$ . By looking at diagonal elements, it is natural to introduce the following

**Axiom 3.** (*Strong Riemann-Roch*)

$$\Delta(F, n) = n \cdot \frac{1}{2} \Delta(F).$$

### 1.3 Duality: Local and Global Pairings

To establish a canonical duality between  $H^0$  and  $H^1$ , we start with the natural pairing

$$\langle \cdot, \cdot \rangle : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1 \subset \mathbb{C}^*$$

defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{v \in S} \langle x_v, y_v \rangle_v, \quad \forall \mathbf{x} = (x_v), \mathbf{y} = (y_v) \in \mathbb{A}.$$

Here

$$\langle x_v, y_v \rangle_v := \prod_{i=1}^n e^{2\pi\sqrt{-1} \cdot \chi_v(x_v, i \cdot y_v, i)}$$

with  $\chi_v := \lambda_v \circ \text{Tr}_{F_v/\mathbb{Q}_v}$ , and

$$\lambda_v := \begin{cases} \mathbb{Q}_v \twoheadrightarrow \mathbb{Q}_v/\mathbb{Z}_v \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}, & v = \mathfrak{p} \in S_{\text{fin}} \\ \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}, & v \in F_\infty. \end{cases}$$

It is well-known that the above global pairing induces a natural isomorphism  $\widehat{\mathbb{A}}^n \simeq \mathbb{A}^n$  (as locally compact groups) which in particular induces an isomorphism  $(F^n)^\perp \simeq F^n$  (as discrete subgroups). On the other hand, by a direct local calculation from the definition ([Ta]), we conclude that, for local pairings  $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$  of finite places  $\mathfrak{p}$ ,  $\mathfrak{a}_{\mathfrak{p}}^\perp \simeq \mathfrak{a}_{\mathfrak{p}}^{-1} \cdot \partial_{\mathfrak{p}}$ , where  $\partial_{\mathfrak{p}}$  denotes the local differential module at  $\mathfrak{p}$  (dual to  $\mathcal{O}_{\mathfrak{p}}$ ). For a fixed uniformizer  $\pi_{\mathfrak{p}}$  of  $\mathcal{O}_{\mathfrak{p}}$ , denote  $\kappa_F$  the idele  $(\pi_{\mathfrak{p}}^{-\text{ord}_{\mathfrak{p}}(\partial_{\mathfrak{p}})}; 1)$ . Since  $H^0$  is insensitive towards local and global units, the cohomology group  $H^0(F, \kappa_F \cdot g^{-1})$  is well-defined. Here, as usual,  $g^{-1}$  denotes the inverse of  $g$ . Consequently, we have the following

**Proposition 3.** (*Topological Duality*) The global pairing  $\langle \cdot, \cdot \rangle : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{C}^*$  induces a natural isomorphism between locally compact groups

$$\widehat{H^1(F, g)} \simeq H^0(F, \kappa_F \otimes g^{-1}).$$

In fact, the above exposes a much stronger relation between  $H^i$ . To understand this properly, let us recall that by Prop. 2, we have a refined isomorphism

$$H^1(F, g) \simeq \left( \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n, \rho(g_\infty) \right) / \Lambda(g).$$

Moreover, the above proof shows that  $\Lambda(g)^\perp \simeq \Lambda(\kappa_F \cdot g^{-1})$ . This completes the proof of the following:

**Theorem 1.** (*Arithmetic Duality*) *There is an isometry of  $\mathcal{O}_F$ -lattices*

$$\widehat{H^1(F, g)} \simeq H^0(F, \kappa_F \cdot g^{-1}).$$

#### 1.4 Riemann-Roch Theorem

With the above duality, and the axioms for Riemann-Roch in mind, to have an Riemann-Roch for our setting, we need to introduce arithmetic counts  $h^0$  and  $h^1$  for discrete groups  $H^0$  and compact groups  $H^1$ . Now the Riemann-Roch in Arakelov theory ([La1,2]) claims that

$$\deg(g) - \frac{n}{2} \log \Delta_F = \chi(F, g) := -\log \left( \text{Vol} \left( \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n, \rho(g_\infty) \right) / \Lambda(g) \right).$$

It is compatible with all axioms above. Therefore,  $h^i$  to be introduced should satisfy

(1) a numerical duality

$$h^1(F, g) = h^0(F, \kappa_F \cdot g^{-1});$$

$$(2) \ h^0(F, g) - h^1(F, g) = -\log \text{Vol} \left( \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n, \rho(g_\infty) \right) / \Lambda(g).$$

•  $h^0(F, g)$ : Count  $H^0(F, g)$  with the weight function

$$\mathbf{e}_\mathbb{A} := \prod_{\mathfrak{p} \in S_{\text{fin}}} \mathbf{1}_{\mathcal{O}_\mathfrak{p}^n} \times \prod_{v \in S_\infty} e^{-\pi \cdot N_v \cdot \|* \|_{\text{can}, v}}.$$

Here  $N_v := [F_v : \mathbb{R}]$  and  $\|* \|_{\text{can}, v}$  denotes the canonical metric at the place  $v$ . That is to say, define the *arithmetical count* of the discrete  $H^0(F, g)$  by

$$\begin{aligned} \#_{\text{Ar}}(H^0(F, g)) &:= \int_{\mathbf{x} \in H^0(F, g)} \mathbf{e}_\mathbb{A}(g \cdot \mathbf{x}) d\mu(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in H^0(F, g)} \prod_{\mathfrak{p} \in S_{\text{fin}}} \mathbf{1}_{\mathcal{O}_\mathfrak{p}^n}(g_\mathfrak{p} \cdot x_\mathfrak{p}) \times \prod_{v \in S_\infty} e^{-\pi N_v \|g_v \cdot x_v \|_{\text{can}, v}}. \end{aligned}$$

(Note that a shift by the multiplicative factor  $g$  is taking place here.)

**Definition 2.** Define the 0-th numerical cohomology  $h^0(F, g)$  of a matrix idele  $g \in GL_n(\mathbb{A})$  by

$$\begin{aligned} h^0(F, G) &:= \log \left( \#_{\text{Ar}}(H^0(F, g)) \right) = \log \left( \sum_{\mathbf{x} \in H^0(F, g)} \mathbf{e}_{\mathbb{A}}(g \cdot \mathbf{x}) \right) \\ &= \log \left( \sum_{\mathbf{x} \in H^0(F, g)} \prod_{\mathfrak{p} \in S_{\text{fin}}} \mathbf{1}_{\mathcal{O}_{\mathfrak{p}}^n}(g_{\mathfrak{p}} \cdot x_{\mathfrak{p}}) \times \prod_{v \in S_{\infty}} e^{-\pi N_v \|g_v \cdot x_v\|_{\text{can}, v}} \right). \end{aligned}$$

•  $h^1(F, g)$ : We start with the following natural

**Axiom 4.** If  $G$  is a discrete or compact group, then arithmetic counts  $\#_{\text{Ar}}$  for  $G$  and its Pontrjagin dual  $\widehat{G}$  coincide. That is,

$$\#_{\text{Ar}}(G) = \#_{\text{Ar}}(\widehat{G}).$$

For our setting, we have the surjection  $\mathbb{A}^n \twoheadrightarrow \mathbb{A}^n / (\mathbb{A}^n(g) + F^n) = H^1(F, g)$  with  $H^1(F, g)$  compact, and hence an injection  $\widehat{H^1(F, g)} \hookrightarrow \widehat{\mathbb{A}^n}$  with  $\widehat{H^1(F, g)}$  discrete. Fix a test function  $f$  on  $\mathbb{A}^n$ , then we obtain its Fourier transform  $\widehat{f}$  using the character  $e^{-2\pi\sqrt{-1}\sum_{v \in S} \chi_v(*)}$  on  $\widehat{\mathbb{A}^n} = \mathbb{A}^n$ . In particular, it makes sense to talk about  $\widehat{f}|_{\widehat{H^1(F, g)}}$ . Now, for the pairing  $(\widehat{H^1(F, g)}, H^1(F, g))$ , using the character  $e^{-2\pi\sqrt{-1}\sum_{v \in S} \chi_v(*)}$  again, we obtain the Fourier transform  $\widetilde{f}$  of  $\widehat{f}|_{\widehat{H^1(F, g)}}$  defined by

$$\widetilde{f}(\eta) := \sum_{\xi \in \widehat{H^1(F, g)}} e^{-2\pi\sqrt{-1}\sum_{v \in S} \chi_v(\xi_v \eta_v)} \widehat{f}(\xi), \quad \forall \eta \in H^1(F, g).$$

Assume the test function  $f$  on  $\mathbb{A}^n$  satisfying

$$f|_{H^0(F, g)} \in L^2(H^0(F, g)), \quad \widehat{f}|_{\widehat{H^1(F, g)}} \in L^2(\widehat{H^1(F, g)}), \quad \widetilde{f} \in L^2(H^1(F, g)).$$

Then we can introduce the arithmetic counts for  $H^0(F, g)$  and  $H^1(F, g)$  with respect to the test function  $f$ . That is to say, by defining

$$\begin{aligned} \#_{\text{Ar}, f}(H^0(F, g)) &:= \int_{H^0(F, g)} |f(\mathbf{x})|^2 d\mathbf{x} = \sum_{\mathbf{x} \in H^0(F, g)} |f(\mathbf{x})|^2; \\ \#_{\text{Ar}, f}(\widehat{H^1(F, g)}) &:= \int_{\widehat{H^1(F, g)}} |\widehat{f}(\mathbf{y})|^2 d\mathbf{y} = \sum_{\mathbf{y} \in \widehat{H^1(F, g)}} |\widehat{f}(\mathbf{y})|^2; \\ \#_{\text{Ar}, f}(H^1(F, g)) &:= \int_{H^1(F, g)} |\widetilde{f}(\mathbf{z})|^2 d\mathbf{z}. \end{aligned}$$

In particular, by the Plancherel formula, we have Axiom 4:

$$\#_{\text{Ar}, f}(\widehat{H^1(F, g)}) = \#_{\text{Ar}, f}(H^1(F, g)).$$

*Remark.* The above procedure can be applied to introduce counts for all discrete subgroups and compact quotient groups of locally compact groups.

To define  $h^1$ , as above, we use the canonical test function

$$\mathbf{e}_{\mathbb{A}}^{\frac{1}{2}} := \prod_{\mathfrak{p} \in S_{\text{fin}}} \mathbf{1}_{\mathcal{O}_{\mathfrak{p}}^n} \times \prod_{v \in S_{\infty}} e^{-\frac{1}{2}\pi N_v \|*\|_{\text{can},v}}$$

on  $\mathbb{A}^n$ . (Recall that being the characteristic function,  $\mathbf{1}_{\mathcal{O}_{\mathfrak{p}}^n} = \mathbf{1}_{\widehat{\mathcal{O}_{\mathfrak{p}}^n}}$ .) We then

have a natural count for  $H^1(F, g)$  using  $\mathbf{e}(\ast) := \left| \widehat{\mathbf{e}_{\mathbb{A}}^{\frac{1}{2}}(\mathbf{g}\ast)} \right|^2$ :

$$\#_{\text{Ar}} H^1(F, g) := \int_{H^1(F, g)} \mathbf{e}(\mathbf{x}) d\mathbf{x}.$$

**Definition 3.** Define the 1-st numerical cohomology  $h^1(F, g)$  for a matrix idele  $g \in GL_n(\mathbb{A})$  by

$$h^1(F, \mathbf{g}) := \log \left( \#_{\text{Ar}}(H^1(F, g)) \right) = \log \left( \int_{H^1(F, \mathbf{g})} \mathbf{e}(\mathbf{x}) d\mu(\mathbf{x}) \right).$$

By a direct calculation ([Ta]), we have

$$\widehat{\mathbf{e}_{\mathbb{A}}} = \prod_{\mathfrak{p} \in S_{\text{fin}}} \mathbf{1}_{(\partial_{\mathfrak{p}}^{-1})^n} \times \prod_{v \in S_{\infty}} e^{-\pi N_v \|*\|_{\text{can},v}}.$$

This, together with our arithmetical duality  $H^1(F, g) \simeq H^0(F, \widehat{\kappa_F \cdot g^{-1}})$ , implies that

$$\#_{\text{Ar}} \widehat{H^1(F, g)} = \sum_{\mathbf{a} \in \widehat{H^1(F, G)}} \widehat{\mathbf{e}_{\mathbb{A}}}(g^{-1}\mathbf{a}) = \#_{\text{Ar}} H^0(F, \kappa_F \cdot g^{-1}),$$

because, in the definition of  $\#_{\text{Ar}} H^0(F, g)$ , a shift by  $g$  is used. But, our counting system satisfies Axiom 4 (due to the Plancherel formula),

$$\#_{\text{Ar}}(H^1(F, g)) = \#_{\text{Ar}} \widehat{H^1(F, g)}.$$

This then proves (1) above, or the same, the numerical duality below. Moreover, from definition of the group  $H^0(F, g)$ , particularly the local condition  $g_v \mathbf{x}_v \in \mathcal{O}_{\mathfrak{p}}^n$ , we see that

$$\#_{\text{Ar}} H^0(F, g) = \sum_{\mathbf{a} \in F^n} \prod_{\mathfrak{p} \in S_{\text{fin}}} \mathbf{1}_{\mathcal{O}_{\mathfrak{p}}^n}(g_{\mathfrak{p}} \mathbf{a}) \times \prod_{v \in S_{\infty}} e^{-\pi N_v \|g_v \mathbf{a}\|_{\text{can},v}} = \sum_{\mathbf{a} \in F^n} \mathbf{e}_{\mathbb{A}}(g \cdot \mathbf{a}).$$

Similarly,

$$\#_{\text{Ar}} H^0(F, \kappa_F \cdot g^{-1}) = \sum_{\mathbf{a} \in F^n} \mathbf{e}_{\mathbb{A}}(\kappa_F \cdot g^{-1} \cdot \mathbf{a}) = \sum_{\mathbf{a} \in F^n} \widehat{\mathbf{e}_{\mathbb{A}}}(g^{-1} \cdot \mathbf{a}).$$

On the other hand, by Tate's Riemann-Roch ([Ta]), obtained from the Poisson summation formula for adelic spaces  $F^n \hookrightarrow \mathbb{A}^n$ , we have

$$\sum_{\mathbf{a} \in F^n} \mathbf{e}_{\mathbb{A}}(g \cdot \mathbf{a}) = \frac{1}{\|\det g\|} \cdot \sum_{\mathbf{a} \in F^n} \widehat{\mathbf{e}}_{\mathbb{A}}(g^{-1} \cdot \mathbf{a}).$$

That is to say,

$$\#_{\text{Ar}} H^0(F, g) = \frac{1}{\|\det g\|} \cdot \#_{\text{Ar}} H^0(F, \kappa_F \cdot g^{-1}).$$

Or equivalently, with the help of the numerical duality (1), we have

$$h^0(F, g) - h^1(F, g) = \chi(F, g).$$

This, together with the Arakelov Riemann-Roch ([La1,2])

$$\chi(F, g) = \deg(g) - \frac{n}{2} \cdot \log \Delta_F,$$

then implies (2) above, and hence completes the proof of the following fundamental:

**Theorem 2.** *For a matrix idele  $g \in GL_n(\mathbb{A})$ , we have*

(i) *(Numerical Duality)*

$$h^1(F, g) = h^0(F, \kappa_F \cdot g^{-1});$$

(ii) *(Arithmetic Riemann-Roch)*

$$h^0(F, g) - h^1(F, g) = \deg(g) - \frac{n}{2} \cdot \log \Delta_F.$$

• **Lattice Version:** In parallel, we have a theory for metrized bundles. Let  $\Lambda$  be an  $\mathcal{O}_F$ -lattice of rank  $n$  in the metrized space  $\left( \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n, \rho \right)$ . Define its *0-th and 1-st cohomological groups* by setting  $H^0(F, \Lambda) = \Lambda$  to be the lattice itself and  $H^1(F, \Lambda) := \left( \prod_{\sigma: \mathbb{R}} \mathbb{R}^n \times \prod_{\tau: \mathbb{R}} \mathbb{C}^n, \rho \right) / \Lambda$  to be its compact quotient. Then, by Theorem 1, we have

**Theorem' 1.** *(Arithmetic Duality) There is a canonical identification*

$$\widehat{H^1(F, \Lambda)} \simeq H^0(F, \omega_F \otimes \Lambda^\vee),$$

where  $\widehat{H^1(F, \Lambda)}$  is the Pontrjagin dual of  $H^1(F, \Lambda)$ , and  $\omega_F$  is the canonical lattice of  $F$  defined by the projective module associated with the inverse of the global differential module  $\partial_F$  of  $\mathcal{O}_F$  together with the standard matrix.



Recall that in algebraic geometry, for a divisor  $D$  of an irreducible algebraic curve over the finite field  $\mathbb{F}_q$ ,

$$q^{\dim H^0(F,D)} = \#H^0(C,D) = \sum_{\mathbf{x} \in H^0(C,D)} \mathbf{1}_{\mathbf{x}}.$$

Motivated by this, to introduce a numerical  $h^0$ , we start with the natural weight function  $e^{-\pi \sum_v N_v \|*\|_{\rho_v}}$ . Thus being discrete, we obtain an arithmetic count for  $H^0(F, \Lambda)$ :

$$\#_{\text{Ar}} H^0(F, \Lambda) := \sum_{\mathbf{x} \in H^0(F, \Lambda)} e^{-\pi \sum_v N_v \|\mathbf{x}\|_{\rho_v}}.$$

This coincides with  $k^0(F, \Lambda)$  from arithmetic effectivity of [GS], and hence yields their well-known  $h^0(F, \Lambda) := \log k^0(F, \Lambda)$ .

As for  $h^1$ , we introduce a counting function  $\mathbf{e}(\mathbf{x})$  on  $H^1(F, \Lambda)$  following the idelic discussion using Fourier transform and  $e^{-\pi \sum_v N_v \|*\|_v}$ . This then gives the arithmetic count of  $H^1(F, \Lambda)$  and the numerical  $h^1(F, \Lambda)$  as follows:

$$\#_{\text{Ar}} H^1(F, \Lambda) := \int_{\mathbf{x} \in H^1(F, \Lambda)} \mathbf{e}(\mathbf{x}) d\mu(\mathbf{x}) \quad \text{and} \quad h^1(F, \Lambda) := \log \#_{\text{Ar}} H^1(F, \Lambda).$$

Consequently, from the standard theory of Fourier analysis for lattices, we see that the topological duality and the Plancherel theorem implies the numerical duality and the Poisson summation theorem (together with the numerical duality and the Arakelov Riemann-Roch) gives the Riemann-Roch:

**Theorem' 2.** *For an  $\mathcal{O}_F$ -lattice  $\Lambda$  of rank  $n$ , we have*

- (i) (Arithmetic Duality)  $h^1(F, \Lambda) = h^0(F, \omega_F \otimes \Lambda^\vee)$ ;
- (ii) (Riemann-Roch)  $h^0(F, \Lambda) - h^1(F, \Lambda) = \deg(\Lambda) - \frac{n}{2} \cdot \log \Delta_F$ .

We end this subsection by drawing reader's attentions to [Bo], [Neu], [Mo], [Se] and [De].

## 1.5 Ampleness and Vanishing Theorem

Two reasons have made arithmetic vanishing theorem appeared difficult. First of all, in current theories, there is no individual arithmetical cohomology  $h^i$  but rather a combined arithmetic Euler characteristic  $\chi$ ; Secondly, even with the genuine arithmetic cohomology groups  $H^i$  and hence  $h^i$ 's, it is impossible to have zero groups  $H^0$  and  $H^1$ : After all, the genuine global  $H^0$  (resp.  $H^1$ ) in the case of number fields  $F$  are discrete groups (resp. compact groups) equal to (resp. dual to) full rank lattices in  $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^n$  which can never become zero.

However, here, we, motivated by [GS] and [Gr], prove the following

**Theorem 3.** (*Vanishing Theorem*) Let  $\mathbf{a} \in GL_1(\mathbb{A})$  be an idele of  $F$ . Then

$$\mathbf{a} \text{ is positive} \quad \Leftrightarrow \quad \lim_{m \rightarrow \infty} h^1(F, \mathbf{a}^m \cdot \mathbf{g}) = 0, \quad \forall \mathbf{g} \in GL_n(\mathbb{A}).$$

By definition, an idele  $\mathbf{a} = (a_p; a_v)$  is called *positive* if

$$\deg(\mathbf{a}) := \sum_{\mathfrak{p}} \text{ord}_{\mathfrak{p}}(a_{\mathfrak{p}}) \cdot N_{\mathfrak{p}} \log p - \sum_v N_v \log |a_v|_v > 0;$$

and  $\mathbf{a} = (a_p; a_v)$  is called *ample* if for each  $g = (g_p; g_v) \in GL_n(\mathbb{A})$ , the unit ball  $B_0(1)$  centered at 0 in the metrized space  $\left( (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^n, \rho(a_{\infty}^m \cdot g_{\infty}) \right)$  contains a basis whose elements are positive at finite places of the lattice  $H^0(F, \mathbf{a}^m \cdot \mathbf{g})$ , for sufficiently large  $m \gg 0$  ([Zh]).

**Theorem 4.** (*Criteria for Ampleness*) Let  $\mathbf{a}$  be an idele of a number field  $F$ . Then the following conditions are equivalent:

- (i)  $\mathbf{a}$  is ample;
- (ii)  $\mathbf{a}$  is positive;
- (iii)  $\lim_{m \rightarrow \infty} h^1(F, \mathbf{a}^m \cdot \mathbf{g}) = 0$  for all  $\mathbf{g} \in GL_n(\mathbb{F})$ .

*Proof.* (i)  $\Rightarrow$  (ii). This is obvious.

(ii)  $\Rightarrow$  (i). This is essentially proved in ([Zh, Thm 2.2]). A technical point here is that for a fixed  $n$  and for all possible  $m, \mathbf{a}$  and  $\mathbf{g}$ ,  $H^0(F, \mathbf{a}^m \cdot \mathbf{g}) \subset F^n$  are full rank lattices in metrized spaces based on (the vector space)  $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^n$ .

Indeed, if  $\mathbf{a}$  is positive, by the Riemann-Roch, for sufficiently large  $m$ ,  $H^0(F, \mathbf{a}^m)$  contains a strictly effective section  $l$ , namely, a section  $l$  satisfying  $\|l\| < 1$ . Thus from what we just said,  $H^0(F, \mathbf{a}^m \cdot \mathbf{g})$  contains a full rank sublattice generated by strictly effective sections  $l^m \cdot e_1, l^m \cdot e_2, \dots, l^m \cdot e_N$ , where  $N := n \cdot [F : \mathbb{Q}]$ , and  $\{e_1, e_2, \dots, e_N\}$  is a fixed basis of  $H^0(F, \mathbf{g})$ . (Here, we view  $l$  as an element of  $F$  and  $e_i$  as elements of  $F^n$ , and the product  $\cdot$  is the scalar multiplication.) This, together with Lemma 1.7 of [Zh], then shows that for sufficiently large  $m$ ,  $H^0(F, \mathbf{a}^m \cdot \mathbf{g})$  is generated by strictly effective sections. This establishes the equivalence of (i) and (ii).

(iii)  $\Rightarrow$  (ii). From (iii), by the duality, we have for  $h^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}) \rightarrow 0$  as  $m \rightarrow \infty$  for  $\mathbf{g} \in GL_n(\mathbb{A})$ . But

$$h^0(F, \mathbf{a}^{-m}) = \log \left( 1 + \sum_{\mathbf{x} \in H^0(F, \mathbf{a}^{-m}), \mathbf{x} \neq 0} e^{-\pi \sum_v N_v \|\mathbf{x}\|_v} \right).$$

Consequently, for any non-zero  $\mathbf{x}(m) \in H^0(F, \mathbf{a}^{-m})$ ,

$$e^{-\pi \sum_v N_v \|\mathbf{x}(m)\|_v} \rightarrow 0, \quad \text{or equivalently, } \|\mathbf{x}(m)\|_v \rightarrow \infty, \quad \text{as } m \rightarrow \infty.$$

Applying this to  $\mathbf{x}(m) := \mathbf{e}^m \in H^0(F, \mathbf{a}^{-m})$ , we conclude that  $\|\mathbf{e}\|_v > 1$ ,  $\forall v$  for all non-zero sections  $\mathbf{e} \in H^0(F, \mathbf{a}^{-1})$ . This then further implies

that  $\deg(\mathbf{a}) \geq 0$ . Indeed, if  $\deg(\mathbf{a}) < 0$  or the same  $\deg(\mathbf{a}^{-1}) > 0$ , by applying the equivalence of (i) and (ii) to  $\mathbf{a}^{-1}$ , (replacing  $\mathbf{a}$  with  $\mathbf{a}^l$  for sufficiently large  $l$  if necessary,) we conclude that  $H^0(F, \mathbf{a}^{-1})$  consists of a  $\mathbb{Z}$ -basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n[F:\mathbb{Q}]}\}$  such that  $\|\mathbf{e}_i\|_v < 1$ , a contradiction.

Moreover, we claim that  $\deg(\mathbf{a}) \neq 0$ . Otherwise, choose  $\mathbf{g}_0$  such that  $\chi(F, \mathbf{g}_0) = \deg(\mathbf{g}_0) - \frac{n}{2} \log \Delta_F > 0$ . Then by the Riemann-Roch,

$$h^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}_0) - h^1(F, \mathbf{a}^{-m} \cdot \mathbf{g}_0) \left( = \deg(\mathbf{g}_0) - \frac{n}{2} \log \Delta_F \right) = \chi(F, \mathbf{g}_0).$$

Consequently,  $\lim_{m \rightarrow \infty} h^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}) \geq \chi(F, \mathbf{g}_0) > 0$ , a contradiction as well. This then implies that  $\deg(\mathbf{a}) > 0$ , namely, (ii).

(ii)  $\Rightarrow$  (iii). By the numerical duality, it suffices to show that

$$\lim_{m \rightarrow \infty} h^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}) = 0 \quad \text{for any fixed } \mathbf{g} \in GL_n(\mathbb{A}).$$

Denote by  $\lambda_1(-m)$  the first Minkowski successive minimum of the lattice  $H^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}) \subset \left( \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \right)^n$ . Then, by Prop. 4.4 of [Gr], it suffices to show that  $\lim_{m \rightarrow \infty} \lambda_1(-m) = +\infty$ , since the  $\mathbb{Z}$ -rank of the  $\mathcal{O}_F$ -lattice  $H^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}) = H^0(F, \mathbf{a}^{-1})^{\otimes m} \otimes H^0(F, \mathbf{g})$  remains to be  $n \cdot [F : \mathbb{Q}]$ .

Suppose, otherwise, that there exists an increasing sequence  $\{m_k\}$  of natural numbers such that  $\lambda_1(-m_k)$  remains bounded. Replacing it with a subsequence if necessary, we may assume that  $\lim_{k \rightarrow \infty} \lambda_1(-m_k) = \lambda_1$ .

We claim that this is impossible. To see this, let us go as follows based on the stability to be introduced in the next section. First of all, denote by

$$\{0\} \subset \Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda_s = H^0(F, \mathbf{g})$$

the Harder-Narasimhan filtration of the  $\mathcal{O}_F$ -lattice  $H^0(F, \mathbf{g})$ . Then the Harder-Narasimhan filtration of the lattice  $H^0(F, \mathbf{a}^{-m} \cdot \mathbf{g})$  is given by

$$\{0\} \subset H^0(F, \mathbf{a}^{-m}) \cdot \Lambda_1 \subset \dots \subset H^0(F, \mathbf{a}^{-m}) \cdot \Lambda_s = H^0(F, \mathbf{a}^{-m} \cdot \mathbf{g}).$$

In particular,  $H^0(F, \mathbf{a}^{-m}) \cdot \Lambda_1$  is, by definition, a maximal semi-stable  $\mathcal{O}_F$ -sublattice of  $H^0(F, \mathbf{a}^{-m} \cdot \mathbf{g})$  with biggest slope and highest rank, and

$$\lim_{m \rightarrow \infty} \deg \left( H^0(F, \mathbf{a}^{-m}) \cdot \Lambda_1 \right) = \lim_{m \rightarrow \infty} \left( -m \cdot \text{rank}(\Lambda_1) \cdot \deg(\mathbf{a}) + \deg(\Lambda_1) \right) \rightarrow -\infty.$$

Secondly, fix a minimal vector  $\mathbf{e}(-m_k) \in H^0(F, \mathbf{a}^{-m_k} \cdot \mathbf{g})$  such that  $\|\mathbf{e}(-m_k)\| = \lambda_1(-m_k)$ , and denote by  $\Lambda_1(-m_k)$  the rank one  $\mathcal{O}_F$ -sublattices of  $H^0(F, \mathbf{a}^{-m_k} \cdot \mathbf{g})$  generated by  $\mathbf{e}(-m_k)$ . Then, by the biggest slope and highest rank property of  $H^0(F, \mathbf{a}^{-m}) \cdot \Lambda_1$  in the the Harder-Narasimhan filtration of  $H^0(F, \mathbf{a}^{-m_k} \cdot \mathbf{g})$ , we conclude that

$$\deg \left( \Lambda_1(-m_k) \right) \leq \frac{\deg \left( H^0(F, \mathbf{a}^{-m_k}) \cdot \Lambda_1 \right)}{\text{rank} \left( H^0(F, \mathbf{a}^{-m_k}) \cdot \Lambda_1 \right)}.$$

Consequently,

$$\lim_{k \rightarrow \infty} \deg(\Lambda_1(-m_k)) = -\infty.$$

Finally, applying Prop 7.1 of [Gr] to the rank one  $\Lambda_1(-m_k)$ , we see that

$$\lambda_1(-m_k) = \|\mathbf{e}(-m_k)\| \geq [F : \mathbb{Q}] \cdot e^{-\frac{2}{[F:\mathbb{Q}]} \cdot \deg(\Lambda_1(-m_k))}.$$

Hence,  $\lambda_1(-m_k)$  are unbounded, a contradiction. This completes the proof.

## 2 Moduli Spaces of Semi-Stable Lattices

### 2.1 Stability

Let  $F$  be a number field with  $\mathcal{O}_F$  the integer ring and  $\Delta_F$  the absolute value of the discriminant. By definition, an  $\mathcal{O}_F$ -lattice  $\Lambda$  is called *semi-stable* if for any  $\mathcal{O}_F$ -sublattice  $\Lambda'$  of  $\Lambda$ , we have

$$\left(\text{Vol}(\Lambda)\right)^{\text{rank}(\Lambda')} \leq \left(\text{Vol}(\Lambda')\right)^{\text{rank}(\Lambda)}, \quad \text{or equivalently,} \quad \mu(\Lambda') \leq \mu(\Lambda),$$

where  $\mu(\Lambda) := \frac{\deg(\Lambda)}{\text{rank}_{\mathcal{O}_F}(\Lambda)}$ ; and a size  $n$  matrix idele  $\mathbf{g} \in \mathbb{G}L_n(\mathbb{A})$  is called *semi-stable* if the associated  $\mathcal{O}_F$ -lattice  $\Lambda(\mathbf{g}) = H^0(F, \mathbf{g})$  is semi-stable. (The above equivalence is established using the Arakelov Riemann-Roch.)

Denote by  $\mathcal{M}(n; d) = \mathcal{M}_F(n; d)$  (resp.  $\mathcal{M}_{\text{Ar}, F}(n; d)$ ) the moduli space of semi-stable lattices of rank  $n$  (resp. semi-stable matrix ideles of size  $n$  and degree  $d$ ). It is known that (see e.g., [Stu], [Gra] and [We3,4,6])

- (i)  $\mathcal{M}_{\text{Ar}, F}(n; d)$  is a closed subset of  $GL_n(\mathbb{A})$ ;
- (ii) the fiber of the natural map  $\mathcal{M}_{\text{Ar}}(n; d) \rightarrow \mathcal{M}(n; d)$  is the compact group  $\prod_{\mathfrak{p} \in S_{\text{fin}}} GL_n(\mathcal{O}_{\mathfrak{p}}) \times \prod_{\sigma: \mathbb{R}} O_n(\mathbb{R}) \times \prod_{\tau: \mathbb{C}} U_n(\mathbb{C})$ ; and
- (iii)  $\mathcal{M}(n; d)$  and hence also  $\mathcal{M}_{\text{Ar}}(n; d)$  are compact.

If  $n = 1$ , the stability condition is automatic, and these compact spaces then coincide with the well-known Arakelov Picard groups  $\text{Pic}^d(F)$  and the idelic class group  $\text{Pic}_{\text{Ar}}^d(F)$ .

### 2.2 Effective Vanishing Theorem

From now on, to avoid duplication, we present only the lattice version of the theory. Since  $h^0$  is a smooth function on the moduli spaces, it is quite natural for us to introduce the *analytic stratifications*  $\mathcal{M}(n, d)^{<(\text{resp. } \leq, >, \geq, =)^T}$  for a fixed real number  $T$ . For example,

$$\mathcal{M}(n, d)^{\leq T} := \left\{ \Lambda \in \mathcal{M}(n, d) : h^0(F, \Lambda) \leq T \right\}.$$

Since  $\mathcal{M}(n, d)$  is compact, we have the following

**Proposition 4.** *For fixed  $n$  and  $d$ , there exist real numbers  $M(n, d)$  and  $m(n, d)$  such that  $\mathcal{M}(n, d)^{<m} = \emptyset$  and  $\mathcal{M}(n, d)^{>M} = \emptyset$  if and only if  $m, M \notin [m(n, d), M(n, d)]$ .*

We call  $m(n, d)$  and  $M(n, d)$  the *minimal* and *maximal* values of  $h^0$  on  $\mathcal{M}(n, d)$  respectively, and  $\delta(n, d) := M(n, d) - m(n, d)$  the *complicity* of  $\mathcal{M}(n, d)$ . Since for any rank one lattice  $\overline{L}$ ,

$$\det(\Lambda^\vee \otimes \overline{L}) = \det(\Lambda)^\vee \otimes (\overline{L})^{\otimes \text{rank} \Lambda},$$

by duality and the Riemann-Roch, we have the following

**Proposition 5.** *(Duality) For extremal values of  $h^0$  on the moduli spaces of semi-stable lattices,*

$$\begin{aligned} M\left(n, n \log \Delta_F - d\right) &= M(n, d) + \left(\frac{n}{2} \log \Delta_F - d\right), \\ m\left(n, n \log \Delta_F - d\right) &= m(n, d) + \left(\frac{n}{2} \log \Delta_F - d\right). \end{aligned}$$

**Theorem 5.** *(Uniform Boundness) For any  $\varepsilon > 0$ , there exists an effectively computable constant  $d_F(n; \varepsilon)$  depending also on  $F$  and  $n$  such that, for all  $d \geq d_F(n; \varepsilon)$*

$$0 < m_F(n; d) - \left(d - \frac{n}{2} \log \Delta_F\right) \leq M_F(n; d) - \left(d - \frac{n}{2} \log \Delta_F\right) < \varepsilon.$$

In particular,

$$\lim_{d \rightarrow \infty} \delta_F(n; d) = 0.$$

*Proof.* Since the moduli spaces are compact, by the Riemann-Roch theorem, it suffices to establish the following:

**Theorem 6.** *(Effective Vanishing Theorem) Let  $\Lambda$  be a rank  $n$  semi-stable  $\mathcal{O}_F$ -lattice.*

(i) *If  $\deg(\Lambda) \leq -[F : \mathbb{Q}] \cdot \frac{n \log n}{2}$ , we have*

$$h^0(\Lambda) \leq \frac{3^{n \cdot [F : \mathbb{Q}]}}{1 - \frac{\log 3}{\pi}} \cdot \left(e^{-\pi \cdot [F : \mathbb{Q}]}\right)^{e^{-\frac{2 \deg(\Lambda)}{n \cdot [F : \mathbb{Q}]}}}.$$

(ii) *If  $\deg(\Lambda) \geq [F : \mathbb{Q}] \cdot \frac{n \log n}{2} + n \log \Delta_F$ , we have*

$$h^1(\Lambda) \leq \frac{3^{n \cdot [F : \mathbb{Q}]}}{1 - \frac{\log 3}{\pi}} \cdot \left(e^{-\pi \cdot [F : \mathbb{Q}] \cdot \Delta_F^{-\frac{2}{[F : \mathbb{Q}]}}}\right)^{e^{\frac{2 \deg(\Lambda)}{n \cdot [F : \mathbb{Q}]}}}.$$

*Remark.* If  $n = 1$ , (i) is proved in [Gr] (see also [GS]): the stability condition in rank one is automatic.

*Proof.* By the numerical duality, it suffice to establish (i). For any non-zero vector  $\mathbf{x}$  of  $\Lambda$ , denote by  $L(\mathbf{x}) = \mathcal{O}_F \cdot \mathbf{x}$ , the  $\mathcal{O}_F$ -lattice generated by  $\mathbf{x}$  in  $\Lambda$ . Then by Lem 7.1 of [Gr], we know that

$$\|\mathbf{x}\|^2 \geq [F : \mathbb{Q}] \cdot e^{-\frac{2}{[F:\mathbb{Q}]} \cdot \deg(L(\mathbf{x}))}.$$

But by the semi-stability of  $\Lambda$ , we have

$$\deg(L(\mathbf{x})) \leq \frac{\deg(\Lambda)}{n} \leq -[F : \mathbb{Q}] \cdot \frac{\log n}{2}.$$

Consequently,  $\|\mathbf{x}\|^2 \geq [F : \mathbb{Q}] \cdot n$ . Therefore, Prop 4.4 of [Gr] can be applied to the  $\mathbb{Z}$ -lattice  $\Lambda$ . Note that from above, (by semi-stability,)

$$\|\mathbf{x}\|^2 \geq [F : \mathbb{Q}] \cdot e^{-\frac{2\deg(\Lambda)}{n \cdot [F:\mathbb{Q}]}}.$$

This implies

$$\lambda_1^2 \geq [F : \mathbb{Q}] \cdot e^{-\frac{2\deg(\Lambda)}{n \cdot [F:\mathbb{Q}]}}.$$

Consequently, by Prop. 4.4 of [Gr], we have

$$\#_{\text{Ar}} H^0(F, \Lambda) \leq 1 + \frac{3^{n \cdot [F:\mathbb{Q}]}}{1 - \frac{\log 3}{\pi}} \cdot e^{-\pi \cdot [F:\mathbb{Q}] \cdot e^{-\frac{2\deg(\Lambda)}{n \cdot [F:\mathbb{Q}]}}.$$

Clearly,  $h^0(F, \Lambda) = \log \#_{\text{Ar}} H^0(F, \Lambda) \leq \#_{\text{Ar}} H^0(F, \Lambda) - 1$ . This completes the proof.

### 2.3 Arithmetic Stratifications

Besides the above analytic stratifications, we can also introduce arithmetic stratifications for these moduli spaces. To explain this, let us assume that  $F_0 \subset F$  is a subfield of  $F$ . Then an  $\mathcal{O}_F$ -lattice  $\Lambda$  may be naturally viewed as an  $\mathcal{O}_{F_0}$ -lattice of rank  $\text{rank}_F(\Lambda) \cdot [F : F_0]$ , which we denote by  $\text{Res}_{F_0}^F(\Lambda)$ . Fix a convex polygon  $g$ . Then as  $\mathcal{O}_{F_0}$ -lattices,  $\text{Res}_{F_0}^F(\Lambda)$  admits a natural Harder-Narasimhan type filtration ([Stu], [Gra], [We3,4]). Denote its associated canonical polygon by  $\bar{g}_{F_0}(\Lambda)$ , and introduce *arithmetical stratifications* by

$$\mathcal{M}_F(n, d)^{\leq F_0 g} := \left\{ \Lambda \in \mathcal{M}_F(n, d) \mid \bar{g}_{F_0}(\Lambda) \leq g \right\}.$$

This is a much more refined stability: Unlike the  $F$ -stability, for elements in  $\text{Pic}^d(F)$ ,  $F_0$ -stability is far from being trivial. In fact, we expect these  $F_0$ -level canonical semi-stable filtration and arithmetical stratifications play key roles in the studies of moduli spaces of semi-stable bundles, say, in finding the arithmetical analogues for results in classical algebraic geometry related to special divisors on curves ([GS], [Gr], [Fr]).

### 3 Non-Abelian Zeta Functions

Let  $F$  be a number field with  $\Delta_F$  the absolute value of the discriminant of  $F$ , and denote  $\mathcal{M}_F(n) := \cup_{d \in \mathbb{R}} \mathcal{M}_F(n, d)$  the moduli space of rank  $n$  semi-stable  $\mathcal{O}_F$ -lattices. The natural Tamagawa measure on  $GL_n(\mathbb{A})$  induces a natural measure on  $\mathcal{M}_F(n)$  which we write as  $d\mu$ .

**Definition 4.** *The rank  $n$  non-abelian zeta function  $\widehat{\zeta}_F(s)$  of  $F$  is the integration*

$$\widehat{\zeta}_{F,n}(s) := \left(\Delta_F^{\frac{n}{2}}\right)^s \cdot \int_{\Lambda \in \mathcal{M}_F(n)} \left(e^{h^0(F, \Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(\Lambda)} d\mu(\Lambda), \quad \text{Re}(s) > 1.$$

**Theorem 7.** (0)  $\widehat{\zeta}_{F,1}(s) \doteq \widehat{\zeta}_F(s)$  the completed Dedekind zeta function;

(1) **(Meromorphic Continuation)**  $\widehat{\zeta}_{F,n}(s)$  is well-defined when  $\text{Re}(s) > 1$  and admits a meromorphic continuation, denoted also by  $\widehat{\zeta}_{F,n}(s)$ , to the whole complex  $s$ -plane;

(2) **(Functional Equation)**  $\widehat{\zeta}_{F,n}(1-s) = \widehat{\zeta}_{F,n}(s)$ ;

(3) **(Singularities & Residues)**  $\widehat{\zeta}_{F,n}(s)$  has two singularities, all simple poles, at  $s = 0, 1$ , with the residues  $\pm \text{Vol}(\mathcal{M}_{F,n}[1])$ , where  $\mathcal{M}_{F,n}[1]$  denotes the moduli space of rank  $n$  semi-stable  $\mathcal{O}_F$ -lattices of volume one.

*Proof.* (0) is essentially due to Iwasawa ([Iw]) and Tate ([Ta]). Write the volume  $T$  (resp.  $\geq T$ , resp.  $\leq T$ ) part of the moduli space  $\mathcal{M}_F(n)$  as  $\mathcal{M}_{F,n}[T]$  (resp.  $\mathcal{M}_{F,n}[\geq T]$ , resp.  $\mathcal{M}_{F,n}[\leq T]$ ), then we have natural decompositions

$$\mathcal{M}_F(n) = \mathcal{M}_{F,n}[\leq T] \cup \mathcal{M}_{F,n}[\geq T] = \cup_{T>0} \mathcal{M}_{F,n}[T]$$

and

$$d\mu = \frac{dT}{T} \cdot d\mu_T,$$

where  $d\mu_T$  denote the natural induced volume form on  $\mathcal{M}_{F,n}[T]$ . Then

$$\begin{aligned} \widehat{\zeta}_{F,n}(s) &= \int_{\Lambda \in \mathcal{M}_F(n)} \left(e^{h^0(F, \Lambda)} - 1\right) \cdot \text{Vol}(\Lambda)^s \cdot d\mu(\Lambda) \\ &= I(s) + A(s) - \alpha(s) \end{aligned}$$

where

$$\begin{aligned} I(s) &:= \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} \left(e^{h^0(F, \Lambda)} - 1\right) \cdot \text{Vol}(\Lambda)^s \cdot d\mu(\Lambda) \\ A(s) &:= \int_{\Lambda \in \mathcal{M}_{F,n}[\leq 1]} e^{h^0(F, \Lambda)} \cdot \text{Vol}(\Lambda)^s \cdot d\mu(\Lambda) \\ \alpha(s) &:= \int_{\Lambda \in \mathcal{M}_{F,n}[\leq 1]} \text{Vol}(\Lambda)^s \cdot d\mu(\Lambda). \end{aligned}$$

**I(s):** By the effective vanishing theorem,  $I(s)$  is holomorphic over the whole complex  $s$ -plane, since it is the integration of  $T$ -exponentially decay function  $(e^{h^0(F,\Lambda)} - 1) \cdot \text{Vol}(\Lambda)^s$  over the space  $\mathcal{M}_{F,n}[\geq 1] = \cup_{T \geq 1} \mathcal{M}_{F,n}[T]$ ;

**A(s):** Note that if  $\Lambda$  is semi-stable, so is  $\kappa_F \otimes \Lambda^\vee$ . Consequently,  $\Lambda \mapsto \kappa_F \otimes \Lambda^\vee$  defines a natural involution on  $\mathcal{M}_F(n)$ , and interchanges  $\mathcal{M}_{F,n}[\geq 1]$  and  $\mathcal{M}_{F,n}[\leq 1]$  by the duality and the Riemann-Roch. Thus

$$\begin{aligned}
A(s) &:= \int_{\Lambda \in \mathcal{M}_{F,n}[\leq 1]} e^{h^0(F,\Lambda)} \cdot e^{-s \cdot \chi(F,\Lambda)} \cdot d\mu(\Lambda) \\
&= \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} e^{h^0(F,\kappa_F \otimes \Lambda^\vee)} \cdot e^{-s \cdot \chi(F,\kappa_F \otimes \Lambda^\vee)} d\mu(\Lambda) \\
&= \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} e^{h^1(F,\Lambda)} \cdot e^{s \cdot \chi(F,\Lambda)} d\mu(\Lambda) \\
&= \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} e^{h^0(F,\Lambda)} \cdot e^{(s-1) \cdot \chi(F,\Lambda)} d\mu(\Lambda) \\
&= \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} e^{h^0(F,\Lambda)} \cdot \text{Vol}(\Lambda)^{1-s} d\mu(\Lambda) \\
&= I(1-s) + \beta(s)
\end{aligned}$$

where

$$\beta(s) := \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} \text{Vol}(\Lambda)^{1-s} \cdot d\mu(\Lambda).$$

Since  $I(1-s)$  is a holomorphic function on  $s$ , it suffices to understand  $\beta(s)$ .

**$\alpha(s)$  and  $\beta(s)$ :** By definition,

$$\begin{aligned}
\alpha(s) &= \int_{\Lambda \in \mathcal{M}_{F,n}[\leq 1]} \text{Vol}(\Lambda)^s \cdot d\mu(\Lambda) \\
&= \int_0^1 \frac{dT}{T} \int_{\Lambda \in \mathcal{M}_{F,n}[T]} \text{Vol}(\Lambda)^s \cdot d_T \mu(\Lambda) \\
&= \int_0^1 \frac{dT}{T} \int_{\Lambda \in \mathcal{M}_{F,n}[T]} T^s \cdot d_T \mu(\Lambda) \\
&= \int_0^1 T^s \frac{dT}{T} \cdot \int_{\Lambda \in \mathcal{M}_{F,n}[T]} d_T \mu(\Lambda)
\end{aligned}$$

Note that there is a natural isomorphism between  $\mathcal{M}_{F,n}[1] \rightarrow \mathcal{M}_{F,n}[T]$ , say by sending  $\Lambda \mapsto \Lambda \otimes \Lambda_0$  for a fixed rank one lattice  $\Lambda_0$  of degree  $\frac{\log T}{n}$ .

$$\begin{aligned}
\alpha(s) &= \int_0^1 T^s \frac{dT}{T} \cdot \text{Vol}(\mathcal{M}_{F,n}[1]) \\
&= \text{Vol}(\mathcal{M}_{F,n}[1]) \cdot \frac{1}{s}.
\end{aligned}$$



Similarly,

$$\begin{aligned}\beta(s) &= \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} \text{Vol}(\Lambda)^{1-s} \cdot d\mu(\Lambda) \\ &= -\text{Vol}(\mathcal{M}_{F,n}[1]) \cdot \frac{1}{1-s}.\end{aligned}$$

All in all, the up shot is

$$\begin{aligned}\widehat{\zeta}_{F,r}(s) &= I(s) + I(1-s) - \alpha(s) + \beta(s) \\ &= I(s) + I(1-s) + \text{Vol}(\mathcal{M}_{F,n}[1]) \cdot \left( \frac{1}{s-1} - \frac{1}{s} \right),\end{aligned}$$

with

$$I(s) = \int_{\Lambda \in \mathcal{M}_{F,n}[\geq 1]} \left( e^{h^0(F,\Lambda)} - 1 \right) \cdot \text{Vol}(\Lambda)^s \cdot d\mu(\Lambda)$$

a holomorphic function of  $s$ . This then proves the theorem.

High rank zetas, the natural non-abelian counterparts of Dedekind zeta functions, are expected to play a central role in the study of non-commutative arithmetic aspects of number fields.

We conclude this paper with the following comments: While our method is a continuation of the classical one due to Chevalley, Weil ([W], see also [Se]), Iwasawa ([Iw]) and Tate ([Ta]), these genuine global arithmetic cohomologies and non-abelian zetas expose new structures for number fields ([We 1-6]). In particular, the study of the so-called abelian parts of our non-abelian zetas is now actively carrying on. For details, please refer to [H], [Ki], [KKS], [Ko], [LS], [Su1,2], [SW] and [We 2-6].<sup>2</sup>

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